Università di Genova
DIBRIS DIPARTIMENTO DI INFORMATICA, BIOINGEGNERIA, ROBOTICA E INGEGNERIA DEI SISTEMI

Towards Abstract and (hopefully) Compositional Operational Reasoning

Francesco Dagnino

T-LADIES kick-off

Who am I?

Postdoc at DIBRIS University of Genoa
Programming Languages research group
Genova Logic Group

Research Interests

- operational semantics and operational reasoning
- type systems (global types, session types, coeffect systems, ...)
- category theory for logics, type theories and programming languages

Reasoning about programs

formal guarantees on the behaviour of programs

- correctness of program transformations/approximations
  program equivalence and distance
- correctness of static/dynamic verification techniques
  type systems, program logics, ...

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Bricks

- formal (mathematical) model of programs: syntax and semantics
- reasoning/proof principles and methods (induction and coinduction, logical relations and predicates, ...)

Operational vs Denotational

two approaches to formal semantics and reasoning

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- programs denote abstract mathematical objects (functions, relations, arrows in a category)
- abstract and quite modular theory
- heavy mathematical tools

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Operational

- describes how a program is executed/evaluated
- lightweight and versatile, wide applicability
- lack of abstract/general results, monolitic, case by case

Operational Reasoning

operational reasoning = (formal) reasoning based on an operational semantics

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several styles of operational semantics

- abstract machines
- small-step semantics
- big-step semantics
- evaluation semantics

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all computer scientists are lazy!

reuse results/techniques already proved/introduced

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The harsh reality

- lack of abstract theories
- results tailored to specific languages
- monolitic development

What can we do?

Operational reasoning in-the-abstract

first steps...

- give a general/abstract definition of operational semantics
- develop general and modular techniques

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In this talk

Part I Abstract Big-Step Semantics

Part II Abstract Evaluation Semantics

Part I

Abstract Big-Step Semantics

An example: call-by-value λ-calculus

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Syntax

$$\begin{array}{rcll} t, s & ::= & x \mid \lambda x.t \mid t\,s & \text{expressions} \\ v, w & ::= & \lambda x.t \mid n & \text{values} \end{array}$$

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judgement: $t \Rightarrow v$
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$$\frac{}{v \Rightarrow v} \qquad \frac{t_1 \Rightarrow \lambda x.s \quad t_2 \Rightarrow v \quad s[v/x] \Rightarrow w}{t_1\,t_2 \Rightarrow w}$$

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- describe the core structure of a big-step semantics
  ⇒ shape of rules
  describing the evaluation process

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- a judgement has shape $c \Rightarrow r$
  configuration c evaluates to result r
- $\mathcal{R}$ is a set of rules of shape

$$\frac{c_1 \Rightarrow r_1 \quad \ldots \quad c_n \Rightarrow r_n}{c \Rightarrow r}$$

where $n \geq 0$ and premises are totally ordered (left-to-right)

Example revisited

$$\frac{t_1 \Rightarrow \lambda x.s \quad t_2 \Rightarrow v \quad s[v/x] \Rightarrow w}{t_1\, t_2 \Rightarrow w}$$

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other strategies

- right-to-left $\dfrac{t_2 \Rightarrow v \quad t_1 \Rightarrow \lambda x.s \quad s[v/x] \Rightarrow w}{t_1\, t_2 \Rightarrow w}$

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other strategies

- right-to-left $\dfrac{t_2 \Rightarrow v \quad t_1 \Rightarrow \lambda x.s \quad s[v/x] \Rightarrow w}{t_1\, t_2 \Rightarrow w}$
- late error detection $\dfrac{t_1 \Rightarrow v_1 \quad t_2 \Rightarrow v_2 \quad v_1 \Rightarrow \lambda x.s \quad s[v_2/x] \Rightarrow w}{t_1\, t_2 \Rightarrow w}$

Results I

An issue

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we show that this distinction is hidden in any big-step semantics

- partial evaluation trees
- explicit wrong computations $c \Rightarrow \text{wrong}$
- explicit non-terminating computations $c \Rightarrow \infty$ (or via traces)

Results II

Proof technique for soundness

A predicate on configurations is sound if
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Semantics with observations

big-step semantics describing also the observable behaviour of a program
general extension to infinite behaviour

Part II

Abstract Evaluation Semantics

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$$t, s ::= \mathbf{val}\ v \mid vw \mid v.1 \mid v.2 \mid t\ \mathbf{to}\ x.s \mid \gamma(v_1, \ldots, v_n)$$

$$\sigma, \tau ::= \zeta \mid \sigma \to \underline{\tau} \mid \sigma \times \tau \mid \mathbf{1}$$

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$\gamma : \sigma_1 \ldots \sigma_n \to \sigma$ is a (parametric) generic effect = atomic effectful operation (e.g., sempling from a distribution, storing a value in a location, ...)

Typing rules

$$\frac{}{\Gamma \vdash x : \sigma}\; x : \sigma \in \Gamma \qquad \frac{}{\Gamma \vdash c : \zeta_c}$$

$$\frac{\Gamma \vdash v_1 : \sigma_1 \quad \ldots \quad \Gamma \vdash v_n : \sigma_n}{\Gamma \vdash \gamma(v_1, \ldots, v_n) : \underline{\sigma}}\; \gamma : \sigma_1 \ldots \sigma_n \to \sigma \qquad \frac{}{\Gamma \vdash \langle\rangle : \mathbf{1}}$$

$$\frac{\Gamma \vdash v : \sigma}{\Gamma \vdash \mathbf{val}\, v : \underline{\sigma}} \qquad \frac{\Gamma \vdash t : \underline{\sigma} \quad \Gamma, x : \sigma \vdash s : \underline{\tau}}{\Gamma \vdash t\, \mathbf{to}\, x.s : \underline{\tau}}$$

$$\frac{\Gamma, x : \sigma \vdash t : \underline{\tau}}{\Gamma \vdash \lambda x.t : \sigma \to \underline{\tau}} \qquad \frac{\Gamma \vdash v : \sigma \to \underline{\tau} \quad \Gamma \vdash w : \sigma}{\Gamma \vdash vw : \underline{\tau}}$$

$$\frac{\Gamma \vdash v : \sigma \quad \Gamma \vdash w : \tau}{\Gamma \vdash \langle v, w\rangle : \sigma \times \tau} \qquad \frac{\Gamma \vdash v : \sigma \times \tau}{\Gamma \vdash v.1 : \underline{\sigma}} \qquad \frac{\Gamma \vdash v : \sigma \times \tau}{\Gamma \vdash v.2 : \underline{\tau}}$$

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$$[\![\mathbf{val}\ v]\!] = \eta(v) \qquad [\![(\lambda x.t)v]\!] = [\![t[v/x]]\!]$$
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it is usually defined as a fixpoint

Syntactic graph

values and computation form a graph $\mathcal{Syn}$ where

- nodes are typing environments Γ, value type σ and computation types $\underline{\sigma}$
- edges from Γ to σ are values s.t. $\Gamma \vdash v : \sigma$
  edges from Γ to $\underline{\sigma}$ are computations s.t. $\Gamma \vdash t : \underline{\sigma}$

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Operational Structure

a $\mathcal{Syn}$-operational struture on $\mathcal{B}$ consists of

- a diagram $S \colon \mathcal{Syn} \to \mathcal{B}$ such that
  $S(x_1 : \sigma_1, \ldots, x_n : \sigma_n) = S(\sigma_1) \times \cdots \times S(\sigma_n)$

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- families of arrows

$$\widehat{\iota} \colon 1 \to S(\mathbf{1}) \qquad \widehat{c} \colon 1 \to S(\zeta)$$

$$\widehat{p_1}_{\sigma,\tau} \colon S(\sigma \times \tau) \to S(\sigma) \qquad \widehat{p_2}_{\sigma,\tau} \colon S(\sigma \times \tau) \to S(\tau)$$

$$\widehat{\beta}_{\sigma,\tau} \colon S(\sigma \to \underline{\tau}) \times S(\sigma) \to S(\tau) \qquad \widehat{\gamma} \colon S(\sigma_1) \times \cdots \times S(\sigma_n) \to T(S(\sigma))$$

$$\widehat{e}_{\sigma} \colon S(\underline{\sigma}) \to T(S(\sigma))$$

satisfying some commutative diagrams

Example: $\mathcal{S}et$-based semantics

- $S(\sigma) = \mathcal{V}_\sigma$ and $S(\underline{\sigma}) = \Lambda_\sigma$
  $S(x_1 : \sigma_1, \ldots, x_n : \sigma_n) = \mathcal{V}_{\sigma_1} \times \cdots \times \mathcal{V}_{\sigma_n}$

- if $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash v : \sigma$ then
  $S(v) = (v_1, \ldots, v_n) \mapsto v[v_1/x_1, \ldots, v_n/x_n]$
- if $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash t : \underline{\sigma}$ then
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- if $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash t : \underline{\sigma}$ then
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- $\widehat{\beta}(\lambda x.t, v) = t[v/x]$
  $\widehat{p_i}(\langle v_1, v_2 \rangle) = v_i$
  $\widehat{e}(t) = [\![t]\!]$

Results

- operational semantics beyond $\mathcal{S}et$ (e.g., stochastic λ calculus in measurable spaces)
- general definition of operational logical relations in terms of fibrations
- proved once and for all the fundamental lemma of operational logical relations
- mathematical foundations of differential logical relations for effectful higher-order distances between programs

References

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A quick comparison

Big-Step Semantics

- more common, based on inference rules, easily understandable
- too weak structure (just sets of rules)

Evaluation Semantics

- rich structure, syntax directed
- easy to implement, formalisation in proof-assistant
- non-termination is difficult
- more sophisticated tools

We are just at the beginning!

- abstract evaluation semantics for arbitrary language
- infinite behaviour in abstract evaluation semantics (delay monad?)
- modularised versions of the two approaches
- composition operators
- language translations, morphisms of operational semantics
- ... suggestions?

Questions?

Thank you!

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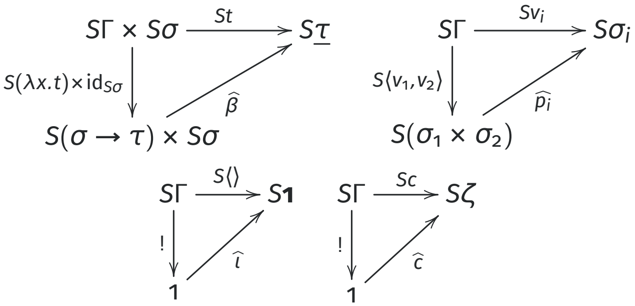
Diagrams for operational structures



Diagrams for operational structures

$$
\begin{array}{ccc}
S\Gamma & \xrightarrow{S(\textbf{val}\ v)} & S\underline{\sigma} \\
{\scriptstyle S(v)}\downarrow & & \downarrow{\scriptstyle \widehat{e}} \\
S\sigma & \xrightarrow[\eta]{} & T(S\sigma)
\end{array}
\qquad
\begin{array}{ccc}
S\Gamma & \xrightarrow{S(\gamma(v_1,\ldots,v_n))} & S(\underline{\sigma}) \\
{\scriptstyle (S(v_1),\ldots,S(v_n))}\downarrow & & \downarrow{\scriptstyle \widehat{e}} \\
S\sigma_1 \times \cdots \times S\sigma_n & \xrightarrow[\widehat{\gamma}]{} & T(S\sigma)
\end{array}
$$

$$
\begin{array}{ccccc}
S\Gamma & & \xrightarrow{S(t\ \textbf{to}\ x.s)} & & S\underline{\tau} \\
{\scriptstyle \langle \mathrm{id}, S(t)\rangle}\downarrow & & & & \downarrow{\scriptstyle \widehat{e}} \\
S\Gamma \times S\underline{\sigma} & \xrightarrow[\mathrm{id}\times\widehat{e}]{} & S\Gamma \times T(S\sigma) & \xrightarrow[\gg\!=(\widehat{e}\circ S(s))]{} & T(S\tau)
\end{array}
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